Evaluate \( n \)th Roots and Use Rational Exponents

**Goal**
- Evaluate \( n \)th roots and study rational exponents.

**VOCABULARY**

- \( n \)th root of \( a \)
- Index of a radical

**REAL \( n \)th ROOTS OF \( a \)**

Let \( n \) be an integer \((n > 1)\) and let \( a \) be a real number.

If \( n \) is an even integer:  
- \( a < 0 \): No real \( n \)th roots.  
- \( a = 0 \): One real \( n \)th root: \( \sqrt[n]{0} = \) ___.
- \( a > 0 \): Two real \( n \)th roots: \( \pm \sqrt[n]{a} = \) ___.

If \( n \) is an odd integer:  
- \( a < 0 \): One real \( n \)th root: \( \sqrt[n]{a} = \) ___.
- \( a = 0 \): One real \( n \)th root: \( \sqrt[n]{0} = \) ___.
- \( a > 0 \): One real \( n \)th root: \( \sqrt[n]{a} = \) ___.

**Example 1**

Find \( n \)th roots

Find the indicated real \( n \)th root(s) of \( a \).

a. \( n = 3, a = -64 \quad b. n = 6, a = 729 \)

**Solution**

a. Because \( n = 3 \) is odd and \( a = -64 \), \(-64\) has ___ \( 0 \), \( -64 \) has ___ \( 0 \). Because \((\_\_\_)^3 = -64\), you can write \( \sqrt[3]{-64} = \) ___ or \((-64)^{1/3} = \) ___.

b. Because \( n = 6 \) is even and \( a = 729 \), \( 729 \) has ___ \( 0 \), \( 729 \) has ___ \( 0 \). Because \( \sqrt[6]{729} = 729 \) and \((\_\_\_)^6 = 729\), you can write \( \pm \sqrt[6]{729} = \) ___ or \( \pm 729^{1/6} = \) ___.

---

160 Lesson 6.1 • Algebra 2 Notetaking Guide
Evaluate \( n \)th Roots and Use Rational Exponents

**Goal**
- Evaluate \( n \)th roots and study rational exponents.

**VOCABULARY**

\( n \)th root of \( a \)

For an integer \( n \) greater than 1, if \( b^n = a \), then \( b \) is an \( n \)th root of \( a \).

Index of a radical

An \( n \)th root of \( a \) is written as \( \sqrt[n]{a} \), where \( n \) is the index of the radical.

**REAL \( n \)th ROOTS OF \( a \)**

Let \( n \) be an integer \((n > 1)\) and let \( a \) be a real number.

If \( n \) is an even integer:
- \( a < 0 \) No real \( n \)th roots.
- \( a = 0 \) One real \( n \)th root: \( \sqrt[n]{0} = 0 \)
- \( a > 0 \) Two real \( n \)th roots:
  - \( a > 0 \) One real \( n \)th root: \( \sqrt[n]{a} = a^{1/n} \)

If \( n \) is an odd integer:
- \( a < 0 \) One real \( n \)th root: \( \sqrt[n]{a} = a^{1/n} \)
- \( a = 0 \) One real \( n \)th root: \( \sqrt[n]{0} = 0 \)
- \( a > 0 \) One real \( n \)th root: \( \sqrt[n]{a} = a^{1/n} \)

**Example 1**

**Find \( n \)th roots**

Find the indicated real \( n \)th root(s) of \( a \).

a. \( n = 3, a = -64 \)  
   b. \( n = 6, a = 729 \)

**Solution**

a. Because \( n = 3 \) is odd and \( a = -64 < 0 \), \(-64 \) has one real cube root. Because \((-4)^3 = -64\), you can write \( \sqrt[3]{-64} = -4 \) or \((-64)^{1/3} = -4 \).

b. Because \( n = 6 \) is even and \( a = 729 > 0 \), \(729 \) has two real sixth roots. Because \( 3^6 = 729 \) and \((-3)^6 = 729\), you can write \( \sqrt[6]{729} = 3 \) or \( \pm 729^{1/6} = \pm 3 \).
Your Notes

**Checkpoint** Find the indicated real \( n \)th roots of \( a \).

1. \( n = 4, a = 256 \)
2. \( n = 3, a = 512 \)

**RATIONAL EXPONENTS**

Let \( a \) be a real number, and let \( m \) and \( n \) be positive integers with \( n > 1 \).

\[ a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a})^m \]

and \( a^{-m/n} = (a^m)^{1/n} = \left(\sqrt[n]{a^m}\right)^n \), \( a \neq 0 \)

**Example 2** Evaluate an expression with rational exponents

Evaluate \( 8^{-4/3} \).

**Solution**

<table>
<thead>
<tr>
<th>Rational Exponent Form</th>
<th>Radical Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 8^{-4/3} = )</td>
<td>( 8^{-4/3} = )</td>
</tr>
<tr>
<td>( \sqrt[3]{8^{-4}} )</td>
<td>( \sqrt[3]{8^{-4}} )</td>
</tr>
<tr>
<td>( (8^{-4/3})^3 = )</td>
<td>( (8^{-4/3})^3 = )</td>
</tr>
<tr>
<td>( 1/ \sqrt[3]{8} = )</td>
<td>( 1/ \sqrt[3]{8} = )</td>
</tr>
<tr>
<td>( 1/ \sqrt[3]{8} = )</td>
<td>( 1/ \sqrt[3]{8} = )</td>
</tr>
<tr>
<td>( 1/ \sqrt[3]{8} = )</td>
<td>( 1/ \sqrt[3]{8} = )</td>
</tr>
<tr>
<td>( 1/ \sqrt[3]{8} = )</td>
<td>( 1/ \sqrt[3]{8} = )</td>
</tr>
</tbody>
</table>

**Example 3** Solve equations using \( n \)th roots

a. \( 2x^6 = 1458 \)
   \[ x^6 = \] \[ x = \]
   \[ x = \] \[ x = \]

b. \( (x + 4)^3 = 12 \)
   \[ x + 4 = \] \[ x = \]
   \[ x = \] \[ x = \]
Checkpoint Find the indicated real $n$th roots of $a$.

1. $n = 4$, $a = 256$
   $$\pm 4$$

2. $n = 3$, $a = 512$
   $$8$$

### RATIONAL EXPONENTS

Let $a$ be a real number, and let $m$ and $n$ be positive integers with $n > 1$.

$$a^{m/n} = (a^{1/n})^m = \left(\sqrt[n]{a}\right)^m$$

and

$$a^{-m/n} = (a^{m/1/n}) = \left(\sqrt[n]{a^m}\right)$$

$$a^{-m/n} = \frac{1}{a^{m/n}} = \frac{1}{(a^{1/n})^m} = \frac{1}{(\sqrt[n]{a})^m}, a \neq 0$$

### Example 2 Evaluate an expression with rational exponents

**Evaluate $8^{-4/3}$.**

**Solution**

**Rational Exponent Form**

$$8^{-4/3} = \frac{1}{8^{4/3}}$$

$$= \frac{1}{(8^{1/3})^4}$$

$$= \frac{1}{2^4}$$

$$= \frac{1}{16}$$

**Radical Form**

$$8^{-4/3} = \frac{1}{8^{4/3}}$$

$$= \frac{1}{(\sqrt[3]{8})^4}$$

$$= \frac{1}{2^4}$$

$$= \frac{1}{16}$$

### Example 3 Solve equations using $n$th roots

**a.** $2x^6 = 1458$

$$x^6 = \frac{729}{2}$$

$$x = \pm \sqrt[6]{729}$$

$$x = \pm 3$$

**b.** $(x + 4)^3 = 12$

$$x + 4 = \sqrt[3]{12}$$

$$x = \sqrt[3]{12} - 4$$

$$x \approx -1.71$$
Animal Population  The population $P$ of a certain animal species after $t$ months can be modeled by $P = C(1.21)^{t/3}$ where $C$ is the initial population. Find the population after 19 months if the initial population was 75.

Solution

$$P = C(1.21)^{t/3}$$  \hspace{1cm} \text{Write model for population.}

$$= \quad \text{Substitute for } C \text{ and } t.$$

$$\approx \quad \text{Use a calculator.}$$

The population of the species is about _____ after 19 months.

Checkpoint  Complete the following exercises.

3. Evaluate $(-125)^{-2/3}$.

4. Solve $(y - 3)^4 = 200$.

5. The volume of a cone is given by $V = \frac{\pi r^2 h}{3}$, where $h$ is the height of the cone and $r$ is the radius. Find the radius of a cone whose volume is 25 cubic inches and whose height is 6 inches.
Animal Population  The population \( P \) of a certain animal species after \( t \) months can be modeled by \( P = C(1.21)^{t/3} \) where \( C \) is the initial population. Find the population after 19 months if the initial population was 75.

Solution

\[
P = C(1.21)^{t/3}
\]

Write model for population.

\[
= 75(1.21)^{19/3}
\]

Substitute for \( C \) and \( t \).

\[
≈ 250.8
\]

Use a calculator.

The population of the species is about \( 251 \) after 19 months.

Checkpoint  Complete the following exercises.

3. Evaluate \((-125)^{-2/3} \).

\[
\frac{1}{25}
\]

4. Solve \((y - 3)^4 = 200 \).

\[
\sqrt[4]{200} + 3 \approx 6.76 \text{ or }\]

\[
-\sqrt[4]{200} + 3 \approx -0.76
\]

5. The volume of a cone is given by \( V = \frac{\pi r^2 h}{3} \), where \( h \) is the height of the cone and \( r \) is the radius. Find the radius of a cone whose volume is 25 cubic inches and whose height is 6 inches.

\( 1.99 \) in.
6.2  Apply Properties of Rational Exponents

Goal  • Simplify expressions involving rational exponents.

VOCABULARY

Simplest form of a radical

Like radicals

PROPERTIES OF RATIONAL EXPONENTS

Let $a$ and $b$ be real numbers and let $m$ and $n$ be rational numbers. The following properties have the same names as those in Lesson 5.1, but now apply to rational exponents.

Property

1. $a^m \cdot a^n = a^{m+n}$  
   $4^{1/2} \cdot 4^{3/2} = 4^{(1/2 + 3/2)}$

2. $(a^m)^n = a^{mn}$  
   $(2^{5/2})^2 = 2^{(5/2 \cdot 2)}$

3. $(ab)^m = a^m b^m$  
   $(16 \cdot 4)^{1/2} = 16^{1/2} \cdot 4^{1/2}$

4. $a^{-m} = \frac{1}{a^m}, a \neq 0$  
   $25^{-1/2} = \frac{1}{25^{1/2}} = ___$

5. $\frac{a^m}{a^n} = a^{m-n}, a \neq 0$  
   $\frac{3^{5/2}}{3^{1/2}} = 3^{(5/2 - 1/2)} = ___$

6. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$  
   $\left(\frac{27}{8}\right)^{1/3} = \frac{27^{1/3}}{8^{1/3}} = ___$
6.2 Apply Properties of Rational Exponents

Goal  • Simplify expressions involving rational exponents.

VOCABULARY

Simplest form of a radical  A radical with index \( n \) is in simplest form if the radicand has no perfect \( n \)th powers as factors and any denominator has been rationalized.

Like radicals  Two radical expressions with the same index and radicand.

PROPERTIES OF RATIONAL EXPONENTS

Let \( a \) and \( b \) be real numbers and let \( m \) and \( n \) be rational numbers. The following properties have the same names as those in Lesson 5.1, but now apply to rational exponents.

Property

1. \( a^m \cdot a^n = a^{m+n} \)

\[ 4^{1/2} \cdot 4^{3/2} = 4^{(1/2 + 3/2)} = 4^2 = 16 \]

2. \( (a^m)^n = a^{mn} \)

\[ (2^{5/2})^2 = 2^{(5/2 \cdot 2)} = 2^5 = 32 \]

3. \( (ab)^m = a^m b^m \)

\[ (16 \cdot 4)^{1/2} = 16^{1/2} \cdot 4^{1/2} = 4 \cdot 2 = 8 \]

4. \( a^{-m} = \frac{1}{a^m}, \ a \neq 0 \)

\[ 25^{-1/2} = \frac{1}{25^{1/2}} = \frac{1}{5} \]

5. \( \frac{a^m}{a^n} = a^{m-n}, \ a \neq 0 \)

\[ \frac{3^{5/2}}{3^{1/2}} = 3^{(5/2 - 1/2)} = 3^2 = 9 \]

6. \( \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, \ b \neq 0 \)

\[ \left(\frac{27}{8}\right)^{1/3} = \frac{27^{1/3}}{8^{1/3}} = \frac{3}{2} \]
Use properties of exponents

Use the properties of rational exponents to simplify the expression.

a. \(9^{1/2} \cdot 9^{3/4} = \ldots\)

b. \((7^{2/3} \cdot 5^{1/6})^3 = \ldots\)

\[= \ldots\]

\[= \ldots\]

c. \(\frac{3^{5/6}}{3^{1/3}} = \ldots\)

d. \(\left(\frac{16^{2/3}}{4^{2/3}}\right)^4 = \ldots\)

Properties of Radicals

Product Property of Radicals

Quotient Property of Radicals

\(\sqrt[n]{a \cdot b} = \ldots\)

\(\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \ldots, b \neq 0\)

Example 2

Use properties of radicals

Use the properties of radicals to simplify the expression.

a. \(\sqrt[3]{27} \cdot \sqrt[3]{9} = \ldots = \ldots = \ldots\) Product property

b. \(\frac{\sqrt[3]{192}}{\sqrt[3]{3}} = \ldots = \ldots = \ldots\) Quotient property

Checkpoint

Simplify the expression.

1. \((6^6 \cdot 5^6)^{-1/6}\)

2. \(\frac{\sqrt{245}}{\sqrt{5}}\)
Use the properties of rational exponents to simplify the expression.

a. \(9^{1/2} \cdot 9^{3/4} = 9^{(1/2 + 3/4)} = 9^{5/4}\)

b. \((7^{2/3} \cdot 5^{1/6})^3 = (7^{2/3})^3 \cdot (5^{1/6})^3 = 7^{(2/3 \cdot 3)} \cdot 5^{(1/6 \cdot 3)} = 7^2 \cdot 5^{1/2} = 49 \cdot 5^{1/2}\)

c. \(\frac{3^{5/6}}{3^{1/3}} = 3^{(5/6 - 1/3)} = 3^{3/6} = 3^{1/2}\)

d. \(\left(\frac{16^{2/3}}{4^{2/3}}\right)^4 = \left(\left(\frac{16}{4}\right)^{2/3}\right)^4 = (4^{2/3})^4 = 4^{(2/3 \cdot 4)} = 4^{8/3}\)

**Example 2** Use properties of radicals

Use the properties of radicals to simplify the expression.

a. \(\sqrt[5]{27} \cdot \sqrt[5]{9} = \sqrt[5]{27 \cdot 9} = \sqrt[5]{243} = 3\) Product property

b. \(\frac{\sqrt[3]{192}}{\sqrt[3]{3}} = \frac{\sqrt[3]{192}}{\sqrt[3]{3}} = \sqrt[3]{64} = 4\) Quotient property

**Checkpoint** Simplify the expression.

1. \((6^6 \cdot 5^6)^{-1/6} = \frac{1}{30}\)

2. \(\frac{\sqrt{245}}{\sqrt{5}} = 7\)
Example 3  Write radicals in simplest form

Write the expression in simplest form.

\[ \sqrt[5]{128} = \text{_______} \quad \text{Factor out perfect fifth power.} \]
\[ = \text{____} \cdot \text{____} \quad \text{Product property} \]
\[ = \text{____} \quad \text{Simplify.} \]

Example 4  Add and subtract like radicals and roots

Simplify the expression.

a. \( 2(12^{2/3}) + 7(12^{2/3}) = \text{________________________} \)

b. \( \sqrt[4]{48} - \sqrt[4]{3} = \text{____} \cdot \text{____} - \text{____} \)
\[ = \text{________________________} \]

Checkpoint  Write the expression in simplest form.

3. \( \sqrt[3]{\frac{5}{9}} \)

4. \( 6\sqrt[4]{6} + 2\sqrt[4]{6} \)

Example 5  Simplify expressions involving variables

Simplify the expression. Assume all variables are positive.

a. \( \sqrt[5]{32x^{15}} = \text{________________________} \)

b. \( (36m^4n^{10})^{1/2} = \text{________________} \)
\[ = \text{________________} \]

c. \( \sqrt[3]{\frac{a^9}{b^6}} = \text{________________} \)

d. \( \frac{42x^4z^7}{6x^{3/2}y^{-3}z^5} = \text{________________} \)
Your Notes

Example 3  Write radicals in simplest form

Write the expression in simplest form.
\[
\begin{align*}
\sqrt[5]{128} &= \sqrt[5]{32 \cdot 4} & \text{Factor out perfect fifth power.} \\
&= \sqrt[5]{32} \cdot \sqrt[5]{4} & \text{Product property} \\
&= 2\sqrt[5]{4} & \text{Simplify.}
\end{align*}
\]

Example 4  Add and subtract like radicals and roots

Simplify the expression.
\[
\begin{align*}
a. \ 2(12^{2/3}) + 7(12^{2/3}) &= (2 + 7)(12^{2/3}) = 9(12^{2/3}) \\
b. \ \sqrt[4]{48} - \sqrt[4]{3} &= \sqrt[4]{16} \cdot \sqrt[4]{3} - \sqrt[4]{3} \\
&= 2\sqrt[4]{3} - \sqrt[4]{3} = (2 - 1)\sqrt[4]{3} = \sqrt[4]{3}
\end{align*}
\]

Checkpoint  Write the expression in simplest form.

3. \[\sqrt[3]{\frac{5}{9}} = \frac{\sqrt[3]{5}}{\sqrt[3]{9}} = \frac{\sqrt[3]{5}}{3}\]

4. \[6\sqrt[4]{6} + 2\sqrt[4]{6} = 8\sqrt[4]{6}\]

Example 5  Simplify expressions involving variables

Simplify the expression. Assume all variables are positive.
\[
\begin{align*}
a. \ \sqrt[3]{32x^{15}} &= \sqrt[5]{2^5 \cdot (x^3)^5} = \sqrt[5]{2^5} \cdot \sqrt[5]{(x^3)^5} = 2x^3 \\
b. \ (36m^4n^{10})^{1/2} &= \sqrt[6]{36^{1/2}(m^4)^{1/2}(n^{10})^{1/2}} \\
&= \sqrt[6]{6m^{4 \cdot 1/2}n^{10 \cdot 1/2}} = \sqrt[6]{6m^2n^5} \\
c. \ \sqrt[3]{\frac{a^9}{b^6}} &= \frac{\sqrt[3]{a^9}}{\sqrt[3]{b^6}} = \frac{\sqrt[3]{a^3}}{\sqrt[3]{b^2}} = \frac{a^3}{b^2} \\
d. \ \frac{42x^4z^7}{6x^{3/2}y^{-3}z^5} &= 7x^{4 - 3/2}y^{-(3)}z^{7 - 5} = 7x^{5/2}y^3z^2
\end{align*}
\]
Your Notes

Example 6  Write variable expressions in simplest form

Write the expression in simplest form. Assume all variables are positive.

\[ \sqrt[4]{\frac{a^2}{b^6}} = \] 
Multiply to make denominator a perfect fourth power.

\[ = \] 
Simplify.

\[ = \] 
Quotient property.

\[ = \] 
Simplify.

Example 7  Add and subtract expressions involving variables

Perform the indicated operation. Assume all variables are positive.

a. \[ 10^{5/y} - 6^{5/y} = \] 

b. \[ 3a^2b^{1/4} + 4a^2b^{1/4} = \]

Checkpoint Simplify the expression. Assume all variables are positive.

5. \[ \sqrt[3]{8x^7y^3z^{11}} \]

6. \[ 7^{3/2}a^5 - a^{3/2}128a^2 \]

Homework

You must multiply the original expression by a form of 1, in this case _____, when simplifying so that the new expression is equivalent.
Example 6  Write variable expressions in simplest form

Write the expression in simplest form. Assume all variables are positive.

\[
\frac{\sqrt{a^2}}{b^6} = \frac{\sqrt{a^2 \cdot b^2}}{b^6} \cdot \frac{\sqrt{b^2}}{b^2} \quad \text{Multiply to make denominator a perfect fourth power.}
\]

\[
= \frac{\sqrt{a^2 b^2}}{b^8} \quad \text{Simplify.}
\]

\[
= \frac{\sqrt[4]{a^2 b^2}}{b^2} \quad \text{Quotient property.}
\]

\[
= \frac{\sqrt[4]{a^2 b^2}}{b^2} \quad \text{Simplify.}
\]

Example 7  Add and subtract expressions involving variables

Perform the indicated operation. Assume all variables are positive.

a. \(10\sqrt[5]{y} - 6\sqrt[5]{y} = (10 - 6)\sqrt[5]{y} = 4\sqrt[5]{y}\)

b. \(3a^2b^{1/4} + 4a^2b^{1/4} = (3 + 4)a^2b^{1/4} = 7a^2b^{1/4}\)

\[\checkmark\] Checkpoint  Simplify the expression. Assume all variables are positive.

5. \(\sqrt[3]{8x^7y^3z^{11}} = \frac{2x^2yz \sqrt[3]{x^2z^2}}{}\)

6. \(7\sqrt[3]{2a^5} - a\sqrt[3]{128a^2} = \frac{3a\sqrt[3]{2a^2}}{}\)

Homework
Goal • Perform operations with functions.

VOCABULARY

Power function

Composition

OPERATIONS ON FUNCTIONS

Let \( f \) and \( g \) be any two functions. A new function \( h \) can be defined by performing any of the four basic operations on \( f \) and \( g \).

<table>
<thead>
<tr>
<th>Operation and Definition</th>
<th>Example: ( f(x) = 3x ), ( g(x) = x + 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Addition</td>
<td>( h(x) = f(x) + g(x) ) \quad h(x) = 3x + (x + 3) \quad = \underline{\text{_______}} )</td>
</tr>
<tr>
<td>Subtraction</td>
<td>( h(x) = f(x) - g(x) ) \quad h(x) = 3x - (x + 3) \quad = \underline{\text{_______}} )</td>
</tr>
<tr>
<td>Multiplication</td>
<td>( h(x) = f(x) \cdot g(x) ) \quad h(x) = 3x(x + 3) \quad = \underline{\text{_______}} )</td>
</tr>
<tr>
<td>Division</td>
<td>( h(x) = \frac{f(x)}{g(x)} ) \quad h(x) = \underline{\text{_______}} )</td>
</tr>
</tbody>
</table>

The domain of \( h \) consists of the \( x \)-values that are in the domains of \underline{\text{_______}}. Additionally, the domain of a quotient does not include \( x \)-values for which \( g(x) = \underline{\text{_______}} \).
Perform Function Operations and Composition

Goal • Perform operations with functions.

VOCABULARY

Power function A function of the form $y = ax^b$ where $a$ is a real number and $b$ is a rational number

Composition The composition of a function $g$ with a function $f$ is $h(x) = g(f(x))$. The domain of $h$ is the set of all $x$-values such that $x$ is in the domain of $f$ and $f(x)$ is in the domain of $g$.

OPERATIONS ON FUNCTIONS

Let $f$ and $g$ be any two functions. A new function $h$ can be defined by performing any of the four basic operations on $f$ and $g$.

<table>
<thead>
<tr>
<th>Operation and Definition</th>
<th>Example: $f(x) = 3x$, $g(x) = x + 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Addition</td>
<td>$h(x) = f(x) + g(x)$</td>
</tr>
<tr>
<td></td>
<td>$h(x) = 3x + (x + 3)$</td>
</tr>
<tr>
<td></td>
<td>$= 4x + 3$</td>
</tr>
<tr>
<td>Subtraction</td>
<td>$h(x) = f(x) - g(x)$</td>
</tr>
<tr>
<td></td>
<td>$h(x) = 3x - (x + 3)$</td>
</tr>
<tr>
<td></td>
<td>$= 2x - 3$</td>
</tr>
<tr>
<td>Multiplication</td>
<td>$h(x) = f(x) \cdot g(x)$</td>
</tr>
<tr>
<td></td>
<td>$h(x) = 3x(x + 3)$</td>
</tr>
<tr>
<td></td>
<td>$= 3x^2 + 9x$</td>
</tr>
<tr>
<td>Division</td>
<td>$h(x) = \frac{f(x)}{g(x)}$</td>
</tr>
<tr>
<td></td>
<td>$h(x) = \frac{3x}{x + 3}$</td>
</tr>
</tbody>
</table>

The domain of $h$ consists of the $x$-values that are in the domains of both $f$ and $g$. Additionally, the domain of a quotient does not include $x$-values for which $g(x) = 0$. 
Example 1  Add and subtract functions

Let \( f(x) = 3x^{1/2} \) and \( g(x) = -5x^{1/2} \). Find the following.

a. \( f(x) + g(x) \)

b. \( f(x) - g(x) \)

c. the domains of \( f + g \) and \( f - g \)

Solution

a. \( f(x) + g(x) = 3x^{1/2} + (-5x^{1/2}) \)

b. \( f(x) - g(x) = 3x^{1/2} - (-5x^{1/2}) \)

c. The functions \( f \) and \( g \) each have the same domain: __________________________. So, the domains of \( f + g \) and \( f - g \) also consist of __________________________.

Example 2  Multiply and divide functions

Let \( f(x) = 7x \) and \( g(x) = x^{1/6} \). Find the following.

a. \( f(x) \cdot g(x) \)

b. \( \frac{f(x)}{g(x)} \)

c. the domains of \( f \cdot g \) and \( \frac{f}{g} \)

Solution

a. \( f(x) \cdot g(x) = (7x)(x^{1/6}) = \)________________________

b. \( \frac{f(x)}{g(x)} = \)________________________

c. The domain of \( f \) consists of __________________________, and the domain of \( g \) consists of __________________________. So, the domain of \( f \cdot g \) consists of __________________________. Because \( g(0) = \)___,

the domain of \( \frac{f}{g} \) is restricted to __________________________.
**Example 1  Add and subtract functions**

Let $f(x) = 3x^{1/2}$ and $g(x) = -5x^{1/2}$. Find the following.

a. $f(x) + g(x)$
b. $f(x) - g(x)$
c. the domains of $f + g$ and $f - g$

**Solution**

a. $f(x) + g(x) = 3x^{1/2} + (-5x^{1/2}) = [3 + (-5)]x^{1/2} = -2x^{1/2}$
b. $f(x) - g(x) = 3x^{1/2} - (-5x^{1/2}) = [3 - (-5)]x^{1/2} = 8x^{1/2}$
c. The functions $f$ and $g$ each have the same domain: all nonnegative real numbers. So, the domains of $f + g$ and $f - g$ also consist of all nonnegative real numbers.

**Example 2  Multiply and divide functions**

Let $f(x) = 7x$ and $g(x) = x^{1/6}$. Find the following.

a. $f(x) \cdot g(x)$
b. \( \frac{f(x)}{g(x)} \)
c. the domains of $f \cdot g$ and $\frac{f}{g}$

**Solution**

a. $f(x) \cdot g(x) = (7x)(x^{1/6}) = 7x^{1 + 1/6} = 7x^{7/6}$
b. \( \frac{f(x)}{g(x)} = \frac{7x}{x^{1/6}} = 7x^{1 - 1/6} = 7x^{5/6} \)
c. The domain of $f$ consists of all real numbers, and the domain of $g$ consists of all nonnegative real numbers. So, the domain of $f \cdot g$ consists of all nonnegative real numbers. Because $g(0) = 0$, the domain of $\frac{f}{g}$ is restricted to all positive real numbers.
1. Let \( f(x) = 5x^{3/2} \) and \( g(x) = -2x^{3/2} \). Find (a) \( f + g \), (b) \( f - g \), (c) \( f \cdot g \), (d) \( \frac{f}{g} \), and (e) the domains.

**Checkpoint** Complete the following exercise.

---

**COMPOSITION OF FUNCTIONS**

The composition of a function \( g \) with a function \( f \) is \( h(x) = \) _______. The domain of \( h \) is the set of all \( x \)-values such that \( x \) is in the domain of \( \) and \( f(x) \) is in the domain of \( \).
1. Let \( f(x) = 5x^{3/2} \) and \( g(x) = -2x^{3/2} \). Find (a) \( f + g \), (b) \( f - g \), (c) \( f \cdot g \), (d) \( \frac{f}{g} \), and (e) the domains.

a. \( 3x^{3/2} \)  

b. \( 7x^{3/2} \)  
c. \( -10x^3 \)  
d. \( -\frac{5}{2} \)  

e. The domain of \( f + g \), \( f - g \), and \( f \cdot g \) is all nonnegative real numbers. The domain of \( \frac{f}{g} \) is all positive real numbers.

**CHECKPOINT**

Complete the following exercise.

**COMPOSITION OF FUNCTIONS**

The composition of a function \( g \) with a function \( f \) is \( h(x) = g(f(x)) \).

The domain of \( h \) is the set of all \( x \)-values such that \( x \) is in the domain of \( f \) and \( f(x) \) is in the domain of \( g \).
Let \( f(x) = 6x^{-1} \) and \( g(x) = 3x + 5 \). Find the following.

a. \( f(g(x)) \)

b. \( g(f(x)) \)

c. \( f(f(x)) \)

d. the domain of each composition

**Solution**

a. \( f(g(x)) = f(3x + 5) = \) 

b. \( g(f(x)) = g(6x^{-1}) = \) 

c. \( f(f(x)) = f(6x^{-1}) = \) 

d. The domain of \( f(g(x)) \) consists of ____________ except \( x = \) ___ because \( g(\) ___ \() = 0 \) is not in the __________. (Note that \( f(0) = \) ___, which is __________.) The domains of \( g(f(x)) \) and \( f(f(x)) \) consist of ____________ except \( x = \) ___, again because ____________.

**Checkpoint** Complete the following exercise.

2. Let \( f(x) = 5x - 4 \) and \( g(x) = 3x^{-1} \). Find (a) \( f(g(x)) \), (b) \( g(f(x)) \), (c) \( f(f(x)) \), and (d) the domain of each composition.
Example 3  Find compositions of functions

Let \( f(x) = 6x^{-1} \) and \( g(x) = 3x + 5 \). Find the following.

a. \( f(g(x)) \)  
   b. \( g(f(x)) \)  
   c. \( f(f(x)) \)

d. the domain of each composition

Solution

a. \( f(g(x)) = f(3x + 5) = \frac{6(3x + 5)^{-1}}{3x + 5} = \frac{6}{3x + 5} \)

b. \( g(f(x)) = g(6x^{-1}) \)
   \[ = 3(6x^{-1}) + 5 = 18x^{-1} + 5 = \frac{18}{x} + 5 \]

c. \( f(f(x)) = f(6x^{-1}) = 6(6x^{-1})^{-1} = \frac{6}{6^{-1}x} = \frac{6}{x} \)

The domain of \( f(g(x)) \) consists of all real numbers except \( x = \frac{5}{3} \) because \( g( \frac{5}{3} ) = 0 \) is not in the domain of \( f \). (Note that \( f(0) = \frac{6}{0} \), which is undefined.) The domains of \( g(f(x)) \) and \( f(f(x)) \) consist of all real numbers except \( x = 0 \), again because \( 0 \) is not in the domain of \( f \).

Checkpoint  Complete the following exercise.

2. Let \( f(x) = 5x - 4 \) and \( g(x) = 3x^{-1} \). Find (a) \( f(g(x)) \), (b) \( g(f(x)) \), (c) \( f(f(x)) \), and (d) the domain of each composition.

a. \( \frac{15}{x} - 4 \)  
   b. \( \frac{3}{5x - 4} \)  
   c. \( 25x - 24 \)

d. The domain of \( f(g(x)) \) and \( f(f(x)) \) is all real numbers except \( x = 0 \). The domain of \( g(f(x)) \) is all real numbers except \( x = \frac{4}{5} \).
6.4 Use Inverse Functions

**Goal**
- Find inverse functions.

**Your Notes**

<table>
<thead>
<tr>
<th>VOCABULARY</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inverse relation</td>
</tr>
<tr>
<td>Inverse function</td>
</tr>
</tbody>
</table>

**Example 1**  *Find an inverse relation*

Find an equation for the inverse of the relation \( y = 7x - 4 \).

\[ y = 7x - 4 \]

Write original equation.

\[ \underline{\text{Switch } x \text{ and } y.} \]

\[ \underline{\text{Add } \text{something} \text{ to each side.}} \]

\[ \underline{\text{Solve for } y. \text{ This is the inverse relation.}} \]

**Example 2**  *Verify that functions are inverses*

Verify that \( f(x) = 7x - 4 \) and \( f^{-1}(x) = \frac{1}{7}x + \frac{4}{7} \) are inverses.

Show that \( f(f^{-1}(x)) = x \).

\[ f(f^{-1}(x)) = f \left( \frac{1}{7}x + \frac{4}{7} \right) \]

\[ = \text{something} \]

\[ = \text{something} \]

\[ = \text{something} \]

Show that \( f^{-1}(f(x)) = x \).

\[ f^{-1}(f(x)) = f^{-1}(7x - 4) \]

\[ = \text{something} \]

\[ = \text{something} \]

\[ = \text{something} \]
## Use Inverse Functions

**Goal**
- Find inverse functions.

### VOCABULARY

**Inverse relation** A relation that interchanges the input and output values of the original relation

**Inverse function** The original relation and its inverse relation whenever both relations are functions

### Example 1  
**Find an inverse relation**

Find an equation for the inverse of the relation \( y = 7x - 4 \).

\[
\begin{align*}
y &= 7x - 4 \\
x &= 7y - 4 \\
x + 4 &= 7y \\
\frac{1}{7}x + \frac{4}{7} &= y
\end{align*}
\]

Write original equation.  
Switch \( x \) and \( y \).  
Add \( 4 \) to each side.  
Solve for \( y \). This is the inverse relation.

### INVERSE FUNCTIONS

Functions \( f \) and \( g \) are inverses of each other provided:

\[ f(g(x)) = x \quad \text{and} \quad g(f(x)) = x \]

The function \( g \) is denoted by \( f^{-1} \), read as “\( f \) inverse.”

### Example 2  
**Verify that functions are inverses**

Verify that \( f(x) = 7x - 4 \) and \( f^{-1}(x) = \frac{1}{7}x + \frac{4}{7} \) are inverses.

Show that \( f(f^{-1}(x)) = x \).  
Show that \( f^{-1}(f(x)) = x \).

\[
\begin{align*}
f(f^{-1}(x)) &= f\left(\frac{1}{7}x + \frac{4}{7}\right) \\
&= 7\left(\frac{1}{7}x + \frac{4}{7}\right) - 4 \\
&= x + 4 - 4 \\
&= x
\end{align*}
\]

\[
\begin{align*}
f^{-1}(f(x)) &= f^{-1}(7x - 4) \\
&= \frac{1}{7}(7x - 4) + \frac{4}{7} \\
&= x - \frac{4}{7} + \frac{4}{7} \\
&= x
\end{align*}
\]
Your Notes

**Checkpoint** Find the inverse of the function. Then verify that your result and the original function are inverses.

1. \( f(x) = -3x + 5 \)

**Example 3** Find the inverse of a power function

Find the inverse of \( f(x) = 4x^2, x \leq 0 \). Then graph \( f \) and \( f^{-1} \).

\[
\begin{align*}
f(x) &= 4x^2 \\
y &= 4x^2
\end{align*}
\]

Write original function.

Replace \( f(x) \) with \( y \).

Switch \( x \) and \( y \).

Divide each side by 4.

Take square roots of each side.

The domain of \( f \) is restricted to negative values of \( x \). So, the range of \( f^{-1} \) must also be restricted to negative values, and therefore the inverse is \( f^{-1}(x) = \)

(If the domain were restricted to \( x \geq 0 \), you would choose \( f^{-1}(x) = \)).

**HORIZONTAL LINE TEST**

The inverse of a function \( f \) is also a function if and only if no horizontal line intersects the graph of \( f \) ______

Function

\[
\begin{array}{c}
\text{Function} \\
\end{array}
\]

Not a function

\[
\begin{array}{c}
\text{Not a function} \\
\end{array}
\]
Your Notes

Checkpoint  Find the inverse of the function. Then verify that your result and the original function are inverses.

1. \( f(x) = -3x + 5 \)
   \[
   f^{-1}(x) = -\frac{1}{3}x + \frac{5}{3}
   \]

Example 3  Find the inverse of a power function

Find the inverse of \( f(x) = 4x^2, x \leq 0 \). Then graph \( f \) and \( f^{-1} \).

\[
\begin{align*}
  f(x) &= 4x^2 \\
  y &= 4x^2 \\
  x &= 4y^2 \\
  \frac{1}{4}x &= y^2 \\
  \pm\frac{1}{2}\sqrt{x} &= y
\end{align*}
\]

The domain of \( f \) is restricted to negative values of \( x \). So, the range of \( f^{-1} \) must also be restricted to negative values, and therefore the inverse is \( f^{-1}(x) = -\frac{1}{2}\sqrt{x} \). (If the domain were restricted to \( x \geq 0 \), you would choose \( f^{-1}(x) = \frac{1}{2}\sqrt{x} \).)

HORIZONTAL LINE TEST

The inverse of a function \( f \) is also a function if and only if no horizontal line intersects the graph of \( f \) more than once.

<table>
<thead>
<tr>
<th>Function</th>
<th>Not a function</th>
</tr>
</thead>
<tbody>
<tr>
<td>![Function Image]</td>
<td>![Not a function Image]</td>
</tr>
</tbody>
</table>
Example 4  \textit{Find the inverse of a cubic function}

Consider the function \( f(x) = \frac{1}{4}x^3 + 3 \). Determine whether the inverse of \( f \) is a function. Then find the inverse.

\textbf{Solution}

Graph the function \( f \). Notice that no \underline{intersection} intersects the graph more than once. So, the inverse of \( f \) is itself a \underline{function}. To find an equation for \( f^{-1} \), complete the following steps.

\[
\begin{align*}
  f(x) &= \frac{1}{4}x^3 + 3 \\
  y &= \frac{1}{4}x^3 + 3 \\
  y &= \frac{1}{4}x^3 + 3
\end{align*}
\]

\underline{Write original function.}

\underline{Replace \( f(x) \) with \( y \).}

\underline{Switch \( x \) and \( y \).}

\underline{Subtract \_\_ from each side.}

\underline{Multiply each side by \_\_.}

\underline{Take cube root of each side.}

The inverse of \( f \) is \( f^{-1}(x) = \underline{\text{equation}} \).

\textbf{Checkpoint} Find the inverse of the function.

2. \( f(x) = 2x^4 + 1 \)

3. \( g(x) = \frac{1}{32}x^5 \)
Example 4  Find the inverse of a cubic function

Consider the function \( f(x) = \frac{1}{4}x^3 + 3 \). Determine whether the inverse of \( f \) is a function. Then find the inverse.

Solution

Graph the function \( f \). Notice that no horizontal line intersects the graph more than once. So, the inverse of \( f \) is itself a function. To find an equation for \( f^{-1} \), complete the following steps.

\[
\begin{align*}
\text{Write original function.} & \quad f(x) = \frac{1}{4}x^3 + 3 \\
\text{Replace } f(x) \text{ with } y. & \quad y = \frac{1}{4}x^3 + 3 \\
\text{Switch } x \text{ and } y. & \quad x = \frac{1}{4}y^3 \\
\text{Subtract 3 from each side.} & \quad x - 3 = \frac{1}{4}y^3 \\
\text{Multiply each side by 4.} & \quad 4x - 12 = y^3 \\
\text{Take cube root of each side.} & \quad \sqrt[3]{4x - 12} = y \\
\text{The inverse of } f \text{ is } f^{-1}(x) = \sqrt[3]{4x - 12}.
\end{align*}
\]

Checkpoint  Find the inverse of the function.

\[
\begin{align*}
2. \quad f(x) &= 2x^4 + 1 \\
& \quad f^{-1}(x) = \sqrt[4]{2x - 1} - \frac{1}{2} \\
3. \quad g(x) &= \frac{1}{32}x^5 \\
& \quad g^{-1}(x) = 2\sqrt[5]{x}
\end{align*}
\]
Goal
- Graph square root and cube root functions.

Your Notes

VOCABULARY
Radical function

PARENT FUNCTIONS FOR SQUARE ROOT AND CUBE ROOT FUNCTIONS

- The parent function for the family of square root functions is \( f(x) = \sqrt{x} \). The domain is \( x \geq 0 \), and the range is \( y \geq 0 \).
- The parent function for the family of cube root functions is \( g(x) = \sqrt[3]{x} \). The domain and range are \( x, y \in \mathbb{R} \).

Example 1

Graph a square root function

Graph \( y = 2\sqrt{x} \), and state the domain and range. Compare the graph with the graph of \( y = \sqrt{x} \).

Solution
Make a table of values and sketch the graph.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
</tr>
</tbody>
</table>

The radicand of a square root is always nonnegative. So, the domain is \( x \geq 0 \). The range is \( y \geq 0 \).

The graph of \( y = 2\sqrt{x} \) is a vertical _______ of the parent graph of \( y = \sqrt{x} \).
Graph Square Root and Cube Root Functions

**Goal**
- Graph square root and cube root functions.

**Your Notes**

**VOCABULARY**

Radical function  A function containing a radical such as \( y = \sqrt[3]{x} \)

**PARENT FUNCTIONS FOR SQUARE ROOT AND CUBE ROOT FUNCTIONS**

- The parent function for the family of square root functions is \( f(x) = \sqrt{x} \). The domain is \( x \geq 0 \), and the range is \( y \geq 0 \).
- The parent function for the family of cube root functions is \( g(x) = \sqrt[3]{x} \). The domain and range are all real numbers.

**Example 1**  **Graph a square root function**

Graph \( y = 2\sqrt{x} \), and state the domain and range. Compare the graph with the graph of \( y = \sqrt{x} \).

**Solution**

Make a table of values and sketch the graph.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>0</td>
<td>2</td>
<td>2.83</td>
<td>3.46</td>
<td>4</td>
</tr>
</tbody>
</table>

The radicand of a square root is always nonnegative. So, the domain is \( x \geq 0 \). The range is \( y \geq 0 \).

The graph of \( y = 2\sqrt{x} \) is a vertical **stretch** of the parent graph of \( y = \sqrt{x} \).
Graph $y = -\frac{1}{2} \sqrt[3]{x}$, and state the domain and range.

Compare the graph with the graph of $y = \sqrt[3]{x}$.

**Solution**

Make a table of values and sketch the graph.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-2$</th>
<th>$-1$</th>
<th>$0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>_____</td>
<td>_____</td>
<td>_____</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>$1$</th>
<th>$2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>_____</td>
<td>_____</td>
</tr>
</tbody>
</table>

The domain and range are ____________________.

The graph of $y = -\frac{1}{2} \sqrt[3]{x}$ is a vertical _______ of the parent graph of $y = \sqrt[3]{x}$ by a factor of _______ followed by a reflection in the x-axis.

**Checkpoint** Graph the function. Then state the domain and range.

1. $y = 2 \sqrt[3]{x}$
2. $y = -2 \sqrt[3]{x}$
Example 2  Graph a cube root function

Graph \( y = -\frac{1}{2}\sqrt[3]{x} \), and state the domain and range.

Compare the graph with the graph of \( y = \sqrt[3]{x} \).

Solution

Make a table of values and sketch the graph.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>0.63</td>
</tr>
<tr>
<td>-1</td>
<td>0.5</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.5</td>
</tr>
<tr>
<td>2</td>
<td>-0.63</td>
</tr>
</tbody>
</table>

The domain and range are all real numbers.

The graph of \( y = -\frac{1}{2}\sqrt[3]{x} \) is a vertical shrink of the parent graph of \( y = \sqrt[3]{x} \) by a factor of \( \frac{1}{2} \) followed by a reflection in the \( x \)-axis.

Checkpoint  Graph the function. Then state the domain and range.

1. \( y = 2\sqrt[3]{x} \)

   The domain and range are all real numbers.

2. \( y = -2\sqrt[3]{x} \)

   Domain \( x \geq 0 \), range \( y \leq 0 \)
Your Notes

GRAPHS OF RADICAL FUNCTIONS
To graph \( y = a\sqrt{x} - h + k \) or \( y = a\sqrt[3]{x} - h + k \), follow these steps:

Step 1 _________ the graph of \( y = a\sqrt{x} \) or \( y = a\sqrt[3]{x} \).
Step 2 Translate the graph ___ units horizontally and ___ units vertically.

Example 3  Graph a translated square root function

Graph \( y = 3\sqrt{x} - 1 + 2 \). Then state the domain and range.

Solution

1. Sketch the graph of \( y = 3\sqrt{x} \). Notice that it begins at the origin and passes through the point (1, ___).

2. Translate the graph. For \( y = 3\sqrt{x} - 1 + 2 \), \( h = ___ \) and \( k = ___ \). So, shift the graph ___________ and ______ ______. The resulting graph starts at (___, ___) and passes through (___, ___).

From the graph, you can see that the domain of the function is ______ and the range of the function is ______.
**Your Notes**

**GRAPHS OF RADICAL FUNCTIONS**

To graph $y = a\sqrt{x} - h + k$ or $y = a\sqrt[3]{x} - h + k$, follow these steps:

**Step 1** Sketch the graph of $y = a\sqrt{x}$ or $y = a\sqrt[3]{x}$.

**Step 2** Translate the graph $h$ units horizontally and $k$ units vertically.

---

**Example 3**

**Graph a translated square root function**

Graph $y = 3\sqrt{x} - 1 + 2$. Then state the domain and range.

**Solution**

1. Sketch the graph of $y = 3\sqrt{x}$. Notice that it begins at the origin and passes through the point (1, 3).

2. Translate the graph. For $y = 3\sqrt{x} - 1 + 2$, $h = 1$ and $k = 2$. So, shift the graph right 1 unit and up 2 units. The resulting graph starts at (1, 2) and passes through (2, 5).

From the graph, you can see that the domain of the function is $x \geq 1$ and the range of the function is $y \geq 2$. 
Example 4  Graph a translated cube root function

Graph \( y = -2\sqrt[3]{x} + 3 - 2 \). Then state the domain and range.

Solution

1. Sketch the graph of \( y = -2\sqrt[3]{x} \).
   Notice that it passes through the origin and the points \((\_, \_\), (\_, \_)\).

2. Note that for \( y = -2\sqrt[3]{x} + 3 - 2 \), \( h = \) ____ and \( k = \) ____. So, shift the graph ___________ and ___________. The resulting graph passes through the points \((\_, \_), (\_, \_), (\_, \_), (\_, \_), \) and \((\_, \_), (\_, \_)\).

From the graph, you can see that the domain and range of the function are both _____________.

Checkpoint  Graph the function. Then state the domain and range.

3. \( y = -\frac{1}{2}\sqrt[3]{x} + 3 + 2 \)

4. \( y = 3\sqrt[3]{x} + 2 \)

Homework
Example 4  Graph a translated cube root function

Graph \( y = -2\sqrt[3]{x} + 3 - 2 \). Then state the domain and range.

Solution

1. Sketch the graph of \( y = -2\sqrt[3]{x} \). Notice that it passes through the origin and the points \((-1, 2)\) and \((1, -2)\).

2. Note that for \( y = -2\sqrt[3]{x} + 3 - 2 \), \( h = \frac{-3}{3} \) and \( k = \frac{-2}{3} \). So, shift the graph **left 3 units** and **down 2 units**. The resulting graph passes through the points \((-4, 0), (-3, -2), \) and \((-2, -4)\).

From the graph, you can see that the domain and range of the function are both **all real numbers**.

**Checkpoint**  Graph the function. Then state the domain and range.

3. \( y = -\frac{1}{2}\sqrt[3]{x} + 3 + 2 \)  
   **domain** \( x \geq -3 \),  
   **range** \( y \leq 2 \)

4. \( y = 3\sqrt[3]{x} + 2 \)  
   The domain and range are all real numbers.
6.6 Solve Radical Equations

**Goal**
- Solve radical equations.

**VOCABULARY**

Radical equation

**SOLVING RADICAL EQUATIONS**

To solve a radical equation, follow these steps:

**Step 1** _______ the radical on one side of the equation, if necessary.

**Step 2** Raise each side of the equation to the same _______ to eliminate the radical and obtain a linear, quadratic, or other polynomial equation.

**Step 3** _______ the polynomial equation using techniques you learned in previous chapters. Check your solution.

**Example 1** Solve a radical equation

Solve \( \sqrt{x + 6} = 3 \).

\[
\sqrt{x + 6} = 3 \quad \text{Write original equation.}
\]

\[
_______ = ___ \quad \text{Square each side to eliminate the radical.}
\]

\[
_______ = ___ \quad \text{Simplify.}
\]

\[
_______ = ___ \quad \text{Subtract ___ from each side.}
\]

The solution is ___. Check this in the original equation.

**Checkpoint** Solve the equation. Check your solution.

1. \( \sqrt[3]{x - 5} + 1 = -1 \)
6.6 Solve Radical Equations

**Goal**
- Solve radical equations.

### VOCABULARY

Radical equation  An equation with a radical that has variables in the radicand

### SOLVING RADICAL EQUATIONS

To solve a radical equation, follow these steps:

**Step 1** Isolate the radical on one side of the equation, if necessary.

**Step 2** Raise each side of the equation to the same power to eliminate the radical and obtain a linear, quadratic, or other polynomial equation.

**Step 3** Solve the polynomial equation using techniques you learned in previous chapters. Check your solution.

#### Example 1  Solve a radical equation

Solve \( \sqrt{x + 6} = 3 \).

Write original equation.

\[
\sqrt{x + 6} = 3
\]

Square each side to eliminate the radical.

\[
(\sqrt{x + 6})^2 = 3^2
\]

Simplify.

\[
x + 6 = 9
\]

Subtract 6 from each side.

\[
x = 3
\]

The solution is 3. Check this in the original equation.

#### Checkpoint  Solve the equation. Check your solution.

1. \( \sqrt[3]{x - 5} + 1 = -1 \)

\[
-3
\]
Example 2  Solve an equation with a rational exponent

\[(3x + 4)^{2/3} = 16\]  Original equation

\[\frac{(3x + 4)^{2/3}}{16} = 1\]  Raise each side to the power \(\frac{3}{2}\).

\[\frac{(3x + 4)^{2/3}}{16} = 1\]  Apply properties of exponents.

\[\frac{(3x + 4)^{2/3}}{16} = 1\]  Simplify.

\[3x + 4 = 1\]  Subtract ___ from each side.

\[3x = -3\]  Divide each side by ___.

The solution is ___. Check this in the original equation.

Example 3  Solve an equation with an extraneous solution

\[x - 2 = \sqrt{x + 10}\]  Original equation

\[\left( x - 2 \right)^2 = (\sqrt{x + 10})^2\]  Square each side.

\[\left( x - 2 \right)^2 = x + 10\]  Expand left side and simplify right side.

\[\left( x - 2 \right)^2 = x + 10\]  Write in standard form.

\[\left( x - 2 \right)^2 = 0\]  Factor.

\[x - 2 = 0 \quad \text{or} \quad x - 2 = 0\]  Zero product property

\[x = ___ \quad \text{or} \quad x = ___\]  Solve for \(x\).

CHECK

Check \(x = ___\).  Check \(x = -___\).

\[x - 2 = \sqrt{x + 10}\]  \[x - 2 = \sqrt{x + 10}\]

\[\left( x - 2 \right)^2 = (\sqrt{x + 10})^2\]  \[\left( x - 2 \right)^2 = (\sqrt{x + 10})^2\]

\[\left( x - 2 \right)^2 = x + 10\]  \[\left( x - 2 \right)^2 = x + 10\]

\[\left( x - 2 \right)^2 = 0\]  \[\left( x - 2 \right)^2 = 0\]

The only solution is ___. (The apparent solution ____ is extraneous.)
Example 2  Solve an equation with a rational exponent

\[(3x + 4)^{2/3} = 16\]

Original equation

\[\left[(3x + 4)^{2/3}\right]^{3/2} = 16^{3/2}\]

Raise each side to the power \( \frac{3}{2} \).

\[3x + 4 = (16^{1/2})^3\]

Apply properties of exponents.

\[3x + 4 = 64\]

Simplify.

\[3x = 60\]

Subtract 4 from each side.

\[x = 20\]

Divide each side by 3.

The solution is 20. Check this in the original equation.

Example 3  Solve an equation with an extraneous solution

\[x - 2 = \sqrt{x + 10}\]

Original equation

\[(x - 2)^2 = (\sqrt{x + 10})^2\]

Square each side.

\[x^2 - 4x + 4 = x + 10\]

Expand left side and simplify right side.

\[x^2 - 5x - 6 = 0\]

Write in standard form.

\[(x - 6)(x + 1) = 0\]

Factor.

\[x - 6 = 0 \text{ or } x + 1 = 0\]

Zero product property

\[x = 6 \text{ or } x = -1\]

Solve for \(x\).

CHECK
Check \(x = 6\).

\[x - 2 = \sqrt{x + 10}\]

\[6 - 2 \stackrel{?}{=} \sqrt{6 + 10}\]

\[4 \stackrel{?}{=} \sqrt{16}\]

\[4 = 4\]

The only solution is 4. (The apparent solution \(-1\) is extraneous.)
Example 4  Solve an equation with two radicals

Solve $\sqrt{x} + 6 + 2 = \sqrt{10} - 3x$.

Write original equation.

\[ \sqrt{x} + 6 + 2 = \sqrt{10} - 3x \]

Square each side.

\[ \underline{\text{equation}} = \underline{\text{equation}} \]

Expand left side and simplify right side.

\[ \underline{\text{equation}} = \underline{\text{equation}} \]

Isolate radical expression.

\[ \underline{\text{equation}} = \underline{\text{equation}} \]

Divide each side by 4.

\[ \underline{\text{equation}} = \underline{\text{equation}} \]

Square each side again.

\[ \underline{\text{equation}} = \underline{\text{equation}} \]

Simplify.

\[ 0 = \underline{\text{equation}} \]

Write in standard form.

\[ 0 = \underline{\text{equation}} \]

Factor.

\[ \underline{\text{equation}} = 0 \text{ or } \underline{\text{equation}} = 0 \]

Zero product property

\[ x = \underline{\text{equation}} \text{ or } x = \underline{\text{equation}} \]

Solve for $x$.

CHECK  Check $x = \underline{\text{equation}}$.

\[ \underline{\text{equation}} \stackrel{?}{=} \underline{\text{equation}} \]

\[ \underline{\text{equation}} \stackrel{?}{=} \underline{\text{equation}} \]

\[ \underline{\text{equation}} = \underline{\text{equation}} \]

Check $x = -\underline{\text{equation}}$.

\[ \underline{\text{equation}} \stackrel{?}{=} \underline{\text{equation}} \]

\[ \underline{\text{equation}} \stackrel{?}{=} \underline{\text{equation}} \]

\[ \underline{\text{equation}} = \underline{\text{equation}} \]

The only solution is $\underline{\text{equation}}$. (The apparent solution $\underline{\text{equation}}$ is extraneous.)
Example 4  Solve an equation with two radicals

Solve $\sqrt{x} + 6 + 2 = \sqrt{10 - 3x}$.

Write original equation.

Square each side.

Expand left side and simplify right side.

Isolate radical expression.

Divide each side by 4.

Square each side again.

Simplify.

Write in standard form.

Factor.

Zero product property

Solve for x.

CHECK  Check $x = 3$.

$\sqrt{3} + 6 + 2 \overset{?}{=} \sqrt{10 - 3(3)}$

$\sqrt{9} + 2 \overset{?}{=} \sqrt{1}$

$5 = 1$

Check $x = -2$.

$\sqrt{-2} + 6 + 2 \overset{?}{=} \sqrt{10 - 3(-2)}$

$\sqrt{4} + 2 \overset{?}{=} \sqrt{16}$

$4 = 4$

The only solution is $-2$. (The apparent solution $3$ is extraneous.)
Checkpoint Solve the equation. Check for extraneous solutions.

2. \(-2x^{4/3} - 21 = -53\)

3. \(x + 2 = \sqrt{2x + 7}\)

4. \(\sqrt{3x + 4} - 1 = \sqrt{x + 5}\)
Checkpoint Solve the equation. Check for extraneous solutions.

2. \(-2x^{4/3} - 21 = -53\)

   \[
   x = \frac{-(-53) + 21}{2 \cdot \frac{4}{3}} = \frac{74}{8/3} = 8
   \]

3. \(x + 2 = \sqrt{2x + 7}\)

   \[
   x = 1
   \]

4. \(\sqrt{3x + 4} - 1 = \sqrt{x + 5}\)

   \[
   x = 4
   \]
**Words to Review**

Give an example of the vocabulary word.

<table>
<thead>
<tr>
<th>$n$th root of $a$</th>
<th>Index of a radical</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simplest form of a radical</td>
<td>Like radicals</td>
</tr>
<tr>
<td>Power function</td>
<td>Composition</td>
</tr>
<tr>
<td>Inverse relation</td>
<td>Inverse function</td>
</tr>
<tr>
<td>Radical function</td>
<td>Radical equation</td>
</tr>
</tbody>
</table>

Review your notes and Chapter 6 by using the Chapter Review on pages 459–461 of your textbook.
Words to Review

Give an example of the vocabulary word.

<table>
<thead>
<tr>
<th>nth root of $a$</th>
<th>Index of a radical</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 is the cube root of 27.</td>
<td>3 is the index of $\sqrt[3]{27}$.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Simplest form of a radical</th>
<th>Like radicals</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3\sqrt{2}x$ is the simplest form of $\sqrt{18}x$.</td>
<td>$7(11^{1/3})$ and $18(11^{1/3})$ are like radicals.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Power function</th>
<th>Composition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x) = 6x^4$</td>
<td>If $f(x) = 3x^2$ and $g(x) = x - 1$, then $g(f(x)) = 3x^2 - 1$.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Inverse relation</th>
<th>Inverse function</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = \frac{1}{6}x + \frac{1}{3}$ is the inverse relation for $y = 6x - 2$.</td>
<td>$f(x) = 3x + 6$ and $f^{-1}(x) = \frac{1}{3}x - 2$ are inverse functions.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Radical function</th>
<th>Radical equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = \sqrt[3]{x} + 5 - 6$</td>
<td>$\sqrt{x} + 7 = -3$</td>
</tr>
</tbody>
</table>

Review your notes and Chapter 6 by using the Chapter Review on pages 459–461 of your textbook.